The beta family at the prime two and modular forms of level three

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Abstract

We use the orientation underlying the Hirzebruch genus of level three to map the beta family at the prime $p = 2$ into the ring of divided congruences. This procedure, which may be thought of as the elliptic greek letter beta construction, yields the f–invariants of this family.

1 Introduction and statement of the results

One of the most fundamental problems in pure mathematics is to understand the structure of the stable homotopy groups of the sphere $\pi_* S^0$, and the Adams–Novikov spectral sequence (ANSS) serves as a powerful tool to attack this problem: Working locally at a fixed prime p, we have

$$\mathrm{E}_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}\left(BP_*, BP_*\right) \Rightarrow \left(\pi_{t-s} S^0\right)_{(p)},$$

and much insight can be gained by resolving its E_2–term into v_n–periodic components [Rav04]. In their seminal paper propagating this chromatic approach, Miller, Ravenel, and Wilson introduced the so-called greek letter map, and computed the 1–line (for all primes) and the 2–line (for odd primes), generated by the alpha and beta families, respectively [MRW77]. The computation of the 2–line for $p = 2$ is due to Shimomura [Shi81]: Let us concentrate on the beta elements at $p = 2$ (there are also products of α's): Starting from certain elements $x_i \in v_2^{-1} BP_*$, $y_i \in v_1^{-1} BP_*$, put

$$a_0 = 1,\ a_1 = 2,\ a_k = 3 \cdot 2^{k-1}\ k \geq 2;$$

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then, for $n \geq 0$, odd $s \geq 1$, $j \geq 1$, $i \geq 0$, subject to the conditions

$$n \geq i,\ 2^i|j,\ j \leq a_{n-i},\ \text{and } j \leq 2^n \text{ if } s = 1 \text{ and } i = 0,$$

the simplest beta elements are given by

$$\beta_{s\cdot 2^n/j,i+1} = \eta\left(x_n^s/2^{i+1}v_1^j\right), \tag{1}$$

where η is the universal greek letter map. In fact, it is sometimes possible to improve divisibility, viz. for n, s, j, and i as above with the additional conditions that

$$n \geq i+1 \geq 2, j = 2 \text{ and } s \geq 3 \text{ if } n = 2,\ \text{and } j \leq a_{n-i-1} \text{ if } n \geq 3,$$

Shimomura defines

$$\beta_{s\cdot 2^n/j,i+2} = \eta\left(x_n^s/2^{i+2}y_i^m\right) \quad \text{where } m = j/2^i, \tag{2}$$

and shows the following relations between the betas in (1) and (2):

(i) $\beta_{s\cdot 2^n/j,i+2} = \beta_{s\cdot 2^n/j,(i+1)+1}$ if $2^{i+1}|j$,

(ii) $2\beta_{s\cdot 2^n/j,i+2} = \beta_{s\cdot 2^n/j,i+1}$.

There are striking number-theoretical patterns lurking in the stable stems which become visible from the chromatic point of view, e.g. the (nowadays) well-known relation between the 1–line and the (denominators of the) Bernoulli numbers. Concerning the 2–line, Behrens has established a precise relation between the beta family for primes $p \geq 5$ and the existence of modular forms satisfying appropriate congruences [Beh09]. On the other hand, using an injection of the 2–line into the ring of divided congruences (tensored with $\mathbb{Q}/\mathbb{Z}$), Laures introduced the f–invariant as a higher analog of the e–invariant [Lau99]. Subsequent work has shown how these approaches can be merged and used to derive the f–invariant of the beta family, albeit still only for $p \geq 5$ [BL08]. A different route has been taken in [HN07], where the f–invariant is represented using Artin–Schreier theory; however, although no longer limted to primes $p \geq 5$, the calculations actually carried out in that reference only take care of two subfamilies (viz. β_t for $p \nmid t$ and $\beta_{s2^n/2^n}$).

Since there has been some progress on our geometrical understanding of the f–invariant through analytical techniques (to an extent where explicit calculations can be done, cf. e.g. [vB08]) it is desirable to have some sort

of 'comparison table'; to this end, we compute the f–invariant of the beta family[1] at the prime $p = 2$. More precisely, we take a look at the following diagram (at the level $N = 3$, i.e. $\Gamma = \Gamma_1(3)$):

$$\begin{tikzcd}
\mathrm{Ext}^0\left(BP_*, BP_*/\left(p^\infty, v_1^\infty\right)\left[v_2^{-1}\right]\right) \arrow[r] \arrow[d] & \mathrm{Ext}^{2,*}\left(BP_*, BP_*\right) \arrow[d] \\
\mathrm{Ext}^0\left(E^\Gamma_*, E^\Gamma_*/\left(p^\infty, v_1^\infty\right)\right) \arrow[r] \arrow[dr, dotted] & \mathrm{Ext}^{2,*}\left(E^\Gamma_*, E^\Gamma_*\right) \arrow[d] \\
 & \underline{\underline{D}}^\Gamma_{*+1} \otimes \mathbb{Q}/\mathbb{Z}
\end{tikzcd} \tag{3}$$

The composition of the vertical arrows on the RHS (which are injective by the results of [Lau99]) accounts for the algebraic portion of the f–invariant, while the upper horizontal arrow produces the beta family. So, in order to compute the f–invariant of a member of this family, we chase its preimage through the composition of the vertical arrow on the LHS and the dotted arrow; put differently, we carry out (a sufficiently large portion of) the elliptic greek letter construction explicitly. The result can be summarized as follows (where, as usual, we abbreviate $\beta_{k/j} = \beta_{k/j,1}$ and $\beta_k = \beta_{k/1}$):

Theorem 1. *The f–invariants of the beta elements of order two are*

(i) *for odd $s \geq 3$:*

$$f\left(\beta_s\right) \equiv \frac{1}{2}\left(\frac{E_1^2 - 1}{4}\right)^s \quad \operatorname{mod} \underline{\underline{D}}^\Gamma_{3s-1}$$

(ii) *for odd $s \geq 1$:*

$$f\left(\beta_{2s/j}\right) \equiv \frac{1}{2}\left(\frac{E_1^2 - 1}{4}\right)^{2s} \quad \operatorname{mod} \underline{\underline{D}}^\Gamma_{6s-j}$$

(iii) *for $l \geq 0$ and odd $s \geq 1$:*

$$\begin{aligned} f\left(\beta_{4s\cdot 2^l/j}\right) &\equiv \frac{1}{2}\left(\frac{E_1^2 - 1}{4}\right)^{4s\cdot 2^l} + \frac{1}{2}\left(\frac{E_1^2 - 1}{4}\right)^{(4s-1)2^l} \quad \operatorname{mod} \underline{\underline{D}}^\Gamma_{12s\cdot 2^l - j} \\ &\equiv \frac{1}{2}\left(\frac{E_1^2 - 1}{4}\right)^{4s\cdot 2^l} \quad \text{if } j \leq 3\cdot 2^l \end{aligned}$$

[1]The situation of products of permanent alpha elements has been studied in [vB09].

Theorem 2. *The f–invariants of the beta elements of higher order are*

(i) *for odd $s \geq 1$:*

$$f\left(\beta_{4s/2,2}\right) \equiv \frac{1}{4}\left(\frac{E_1^2-1}{4}\right)^{4s} \quad \mathrm{mod}\ \underline{\underline{D}}^{\Gamma}_{12s-2}$$

(ii) *for $l \geq 0$, $i \geq 1$, $j = m \cdot 2^i \leq a_{l+2}$, odd $s \geq 1$, and mod $\underline{\underline{D}}^{\Gamma}_{3s\cdot 2^{l+i+2}-j}$:*

$$\begin{aligned} f\left(\beta_{s\cdot 2^{l+i+2}/j,i+1}\right) &\equiv \frac{1}{2^{i+1}}\left(\frac{E_1^2-1}{4}\right)^{s\cdot 2^{l+i+2}} + \frac{1}{2}\left(\frac{E_1^2-1}{4}\right)^{\left(s\cdot 2^{i+2}-1\right)2^l} \\ &\equiv \frac{1}{2^{i+1}}\left(\frac{E_1^2-1}{4}\right)^{s\cdot 2^{l+i+2}} \qquad \text{if } j \leq 3\cdot 2^l \end{aligned}$$

(iii) *for $k \geq 2$:*

$$f\left(\beta_{4k/2,3}\right) \equiv \frac{1+4k}{8}\left(\frac{E_1^2-1}{4}\right)^{4k} \quad \mathrm{mod}\ \underline{\underline{D}}^{\Gamma}_{12k-2}$$

(iv) *for $l \geq 0$, $i \geq 1$, $j = m \cdot 2^i \leq a_{l+2}$, odd $s \geq 1$, and mod $\underline{\underline{D}}^{\Gamma}_{3s\cdot 2^{l+i+3}-j}$:*

$$\begin{aligned} f\left(\beta_{s\cdot 2^{l+i+3}/j,i+2}\right) &\equiv \frac{1}{2^{i+2}}\left(\frac{E_1^2-1}{4}\right)^{s\cdot 2^{l+i+3}} + \frac{1}{2}\left(\frac{E_1^2-1}{4}\right)^{\left(s\cdot 2^{i+3}-1\right)2^l} \\ &\equiv \frac{1}{2^{i+2}}\left(\frac{E_1^2-1}{4}\right)^{s\cdot 2^{l+i+3}} \qquad \text{if } j \leq 3\cdot 2^l \end{aligned}$$

The proof presented in the following section turns out to be a pretty much straightforward calculation: After a brief recollection of the relevant definitions, we study the image (under the orientation underlying the Hirzebruch genus) of the elements x_i and y_i occurring in the definition of the beta elements. Then, we are going to sketch our approach to the argument given in [BL08, section 4], i.e. we explain how to carry out the greek letter map on the level of (holomorphic) modular forms. The final step consists of performing this computation explicitly.

2 Proof of the Theorems

2.1 Preliminaries

Following [Lau99], we consider the congruence subgroup $\Gamma = \Gamma_1(N)$ for a fixed level $N > 1$, set $\mathbb{Z}^\Gamma = \mathbb{Z}[\zeta_N, 1/N]$ and denote by M_*^Γ the graded ring of modular forms w.r.t. Γ which expand integrally, i.e. which lie in $\mathbb{Z}^\Gamma[\![q]\!]$. The ring of *divided congruences* D^Γ consists of those rational combinations of modular forms which expand integrally; this ring can be filtered by setting

$$D_k^\Gamma = \left\{ f = \textstyle\sum_{i=0}^k f_i \;\middle|\; f_i \in M_i^\Gamma \otimes \mathbb{Q},\ f \in \mathbb{Z}^\Gamma[\![q]\!] \right\}.$$

Finally, we introduce

$$\underline{\underline{D}}_k^\Gamma = D_k^\Gamma + M_0^\Gamma \otimes \mathbb{Q} + M_k^\Gamma \otimes \mathbb{Q}.$$

Now, if Ell^Γ denotes the complex oriented elliptic cohomology theory associated to the universal curve over the ring of modular forms w.r.t. Γ, the composite

$$\mathrm{E}_2^{2,2k+2}[MU] \to \mathrm{E}_2^{2,2k+2}[Ell^\Gamma] \to \underline{\underline{D}}_{k+1}^\Gamma \otimes \mathbb{Q}/\mathbb{Z}$$

is injective (away from primes dividing the level N) [Lau99]. Henceforth, we fix $p = 2$ and $N = 3$. Thus we have

$$M_*^\Gamma = \mathbb{Z}^\Gamma[E_1, E_3],$$

where

$$E_1 = 1 + 6 \sum_{n=1}^\infty \sum_{d|n} (\frac{d}{3})\ q^n,$$

$$E_3 = 1 - 9 \sum_{n=1}^\infty \sum_{d|n} (\frac{d}{3}) d^2\ q^n$$

are the odd Eisenstein series of the indicated weight at the level $N = 3$ (and $(\frac{d}{3})$ denotes the Legendre symbol). Furthermore, the following basic congruence can be read off of the q-expansions:

$$E_3 - 1 \equiv \frac{E_1^2 - 1}{4} \mod 2D_3^\Gamma. \tag{4}$$

2.2 The image under the orientation

Recall that the power series associated to the Hirzebruch elliptic genus of level three may be expressed as (see e.g. [vB08])

$$Q(x) = \exp\left(3\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}G^{*}_{2n}(\tau) - 2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}G^{(-\omega)}_{2k+1}(\tau)\right) \in M^{\Gamma}_{*}\otimes\mathbb{Q}[\![x]\!]$$

where $\omega = 2\pi i/3$ and

$$\begin{aligned} G^{*}_{2n}(\tau) &= G_{2n}(\tau) - 3^{2n-1}G_{2n}(3\tau), \\ G^{(-\omega)}_{2k+1}(\tau) &= \frac{e^{\omega}-e^{-\omega}}{2}3^{2k}\frac{B_{2k+1}(1/3)}{2k+1}E^{\Gamma_1(3)}_{2k+1}(\tau). \end{aligned}$$

The first few terms of this power series, when expressed in terms of E_1 and E_3, i.e. the generators of M^{Γ}_{*}, read

$$\begin{aligned} Ell^{\Gamma_1(3)}(x) = 1 &+ \frac{iE_1}{2\sqrt{3}}x + \frac{E_1^2}{12}x^2 + \frac{iE_1^3 - iE_3}{18\sqrt{3}}x^3 + \frac{13E_1^4 - 16E_1E_3}{2160}x^4 \\ &+ \frac{iE_1^2(E_1^3 - E_3)}{216\sqrt{3}}x^5 + \frac{121E_1^6 - 152E_1^3E_3 + 40E_3^2}{272160}x^6 \\ &+ \frac{iE_1}{\sqrt{3}}\frac{7E_1^6 - 11E_1^3E_3 + 4E_3^2}{19440}x^7 + O(x^8) \end{aligned} \tag{5}$$

The genus of the following complex projective spaces is readily evaluated:

$$\begin{aligned} w_1 = \phi(\mathbb{CP}^1) &= \frac{i}{\sqrt{3}}E_1, \\ w_3 = \phi(\mathbb{CP}^3) &= \frac{i}{\sqrt{3}}\frac{5E_1^3 - 2E_3}{9}, \\ w_7 = \phi(\mathbb{CP}^7) &= \frac{i}{\sqrt{3}}\frac{70E_1^4E_3 - 14E_1E_3^2 - 65E_1^7}{243}. \end{aligned}$$

As is well-known, underlying this genus is the complex orientation of the cohomology theory Ell^{Γ}, i.e.

$$\phi : MU \to Ell^{\Gamma}$$

and we can compute the images of the Hazewinkel generators [Rav04, Appendix A2] at the prime $p = 2$, which we still denote by v_i:

$$v_1 = w_1 = \frac{i}{\sqrt{3}}E_1$$

$$v_2 = \frac{w_3 - w_1^3}{2} = \frac{i}{\sqrt{3}} \frac{4E_1^3 - E_3}{9},$$

$$v_3 = \frac{w_7}{4} - \frac{w_1^7 + w_1 w_3^2}{8} = \frac{iE_1}{\sqrt{3}} \frac{5E_1^3 E_3 - E_3^2 - 4E_1^6}{81}.$$

In particular, we see that v_3 is decomposable:

$$\begin{aligned} v_3 &= \frac{iE_1}{\sqrt{3}} \left(\frac{4E_1^3 E_3 - E_3^2}{81} - \frac{4E_1^6 - E_1^3 E_3}{81} \right) \\ &= \frac{iE_1}{\sqrt{3}} \left(\frac{i}{\sqrt{3}} \frac{4E_1^3 - E_3}{9} \right) \left(-\frac{i}{3\sqrt{3}} \left(E_3 - E_1^3 \right) \right) \\ &= 3v_1 v_2 \left(v_2 + v_1^3 \right) \end{aligned} \tag{6}$$

Plugging (6) into the definitions of the x_i (considered in $v_2^{-1} M_*^\Gamma$) yields

$$\begin{aligned} x_0 &= v_2 \\ x_1 &= v_2^2 - v_1^2 v_2^{-1} v_3 = v_2^2 - 3v_1^3 \left(v_2 + v_1^3 \right) \\ x_2 &= x_1^2 - v_1^3 v_2^3 - v_1^5 v_3 = v_2^4 - 7v_1^3 v_2^3 + 15 v_1^9 v_2 + 9 v_1^{12} \\ x_i &= x_{i-1}^2 \quad i \geq 3, \end{aligned} \tag{7}$$

showing that the x_i are actually holomorphic. On the other hand, unless $i = 0$, this is not true for the $y_i \in v_1^{-1} M_*^\Gamma$, which read:

$$\begin{aligned} y_0 &= v_1 \\ y_1 &= v_1^2 - 4v_1^{-1} v_2 \\ y_i &= y_{i-1}^2 \quad i \geq 2. \end{aligned} \tag{8}$$

However, for $i \geq 1$ and $m \geq 1$, we may introduce

$$z_{i,m} = v_1^{m \cdot 2^i} - m \cdot 2^{i+1} v_1^{m \cdot 2^i - 3} v_2, \tag{9}$$

which are holomorphic for $m \cdot 2^i \geq 4$ and satisfy

$$\begin{aligned} z_{i,m} &\equiv y_i^m \mod 2^{i+2} v_1^{-1} M_*^\Gamma \\ &\equiv 1 \mod 2^{i+2} \mathbb{Z}^\Gamma [\![q]\!], \end{aligned}$$

the second line being an immediate consequence of (4).

2.3 Determining 'elliptic' beta elements

Requiring $p > 3$ and working with the full modular group, Behrens and Laures have shown in [BL08, section 4] how an element in $H^0\left(M_*/\left(p^\infty, E_{p-1}^\infty\right)\right)$ gives rise to an element in $D\otimes\mathbb{Q}/D[\frac{1}{6}]+M_k\otimes\mathbb{Q}+\mathbb{Q}$; clearly, the other primes can be treated analogously by working with a smaller congruence subgroup. Let us rephrase their argument in a language closer to the original formulation of the greek letter construction:

Still working at the prime $p = 2$ and the level $N = 3$, we choose a (holomorphic) modular form $\mu \in M^\Gamma_{|\mu|}$ and a pair of positive integers (i_0, i_1) such that

$$\mu^{i_1} \equiv 1 \mod 2^{i_0} D^\Gamma_{i_1|\mu|}; \tag{10}$$

in particular, this ensures that $(2^{i_0}, \mu^{i_1})$ is regular on M^Γ_*.

Now, given a modular form $\tilde{\varphi}_t \in M^\Gamma_t$, we can use the natural inclusion

$$M^\Gamma_t \hookrightarrow D^\Gamma_t$$

and ask whether $\tilde{\varphi}_t$ satisfies

$$\tilde{\varphi}_t \equiv \mu^{i_1}\varphi_{t/i_1|\mu|,i_0} \mod 2^{i_0} D^\Gamma_t \tag{11}$$

for some

$$\varphi_{t/i_1|\mu|,i_0} \in D^\Gamma_{t-i_1|\mu|}/2^{i_0} D^\Gamma_{t-i_1|\mu|}$$

Let us call a modular form satisfying (11) *invariant mod* $(2^{i_0}, \mu^{i_1})$. Moreover, we have the obvious composition

$$\underline{\underline{(\ \cdot\ )}} : D^\Gamma_k/2^{i_0} D^\Gamma_k \cong D^\Gamma_k \otimes \mathbb{Z}/2^{i_0} \to D^\Gamma_k \otimes \mathbb{Q}/\mathbb{Z} \to \underline{\underline{D}}^\Gamma_k \otimes \mathbb{Q}/\mathbb{Z},$$

$$\varphi_k \mapsto \underline{\underline{\varphi}}_k$$

Then it is easy to see that, for an invariant modular form $\tilde{\varphi}_t$, the assignment

$$\tilde{\varphi}_t \mapsto \underline{\underline{\varphi}}_{t/i_1|\mu|,i_0}$$

depends only on the reduction $\varphi_t \equiv \tilde{\varphi}_t \mod (2^{i_0}, \mu^{i_1})$, hence descends to a well-defined map

$$\ker\left(M^\Gamma_t/\left(2^{i_0}, \mu^{i_1}\right) \to D^\Gamma_t/\left(2^{i_0}, \mu^{i_1}\right)\right) \longrightarrow \underline{\underline{D}}^\Gamma_{t-i_1|\mu|} \otimes \mathbb{Q}/\mathbb{Z} \tag{12}$$

which we may think of as the *'elliptic' greek letter beta map* and which corresponds to the dotted arrow in (3).

Remark 1. By removing the constant term of the q-expansion, we obtain another map

$$d\colon M_t^\Gamma \to D_t^\Gamma, \quad d\left(\tilde{\varphi}_t\right) = \tilde{\varphi} - q^0\left(\tilde{\varphi}_t\right)$$

that might look like a more natural choice w.r.t. which invariance should be defined (cf. [BL08]). However, we have $q^0(\varphi) \equiv \mu^{i_1} q^0(\varphi) \mod 2^{i_0} D_t^\Gamma$, hence both choices are equivalent (up to the shift of $\varphi_{t/i_1|\mu|,i_0}$ by the constant $q^0\left(\tilde{\varphi}_t\right)$, which maps to zero in $\underline{\underline{D}}_k^\Gamma \otimes \mathbb{Q}/\mathbb{Z}$).

2.4 Explicit computations

In this subsection, we are going to compute the effect of the map (12) on the preimage of Shimomura's beta elements; the ones defined by (1) are dealt with easily, since $\left(2^{i+1}, v_1^j\right)$ is regular on M_*^Γ provided that $j = m \cdot 2^i$; moreover, for $k \geq 0$ this implies:

$$\left(\frac{E_1^2 - 1}{4}\right)^k \equiv v_1^j \left(\frac{E_1^2 - 1}{4}\right)^k \mod 2^{i+1} D_{2k+j}^\Gamma \tag{13}$$

Furthermore, the following two results are useful:

Lemma 1. *For $i \geq 0$, $l \geq 0$, $m \cdot 2^i = j \leq 6 \cdot 2^l$ we have:*

$$E_3^{s \cdot 2^{l+i+2}} \equiv \left(\frac{E_1^2 - 1}{4}\right)^{s \cdot 2^{l+i+2}} \mod 2^{i+1} D_{12s \cdot 2^{l+i}}^\Gamma + v_1^j \cdot M_{12s \cdot 2^{l+i} - j}^\Gamma$$

Proof. It is easy to see that for $l \geq 0$ and $i \geq 0$, we have

$$E_3^{2^{l+i+2}} \equiv \left(E_3 - v_1^3\right)^{2^{l+i+2}} + 2^{i+1} \left(v_1^6 E_3^2\right)^{2^l} E_3^{2^{l+2}\left(2^i - 1\right)} \mod \left(2^{i+2}, v_1^{12 \cdot 2^l}\right), \tag{14}$$

and the basic congruence (4) implies

$$\left(E_3 - v_1^3\right)^{2^k} \equiv \left(\frac{E_1^2 - 1}{4}\right)^{2^k} \mod 2^{k+1} D_{3 \cdot 2^k}^\Gamma \tag{15}$$

□

Lemma 2. *For $i \geq 0$, $l \geq 0$, $1 \leq j \leq 6 \cdot 2^l$ we have:*

$$\begin{aligned} E_3^{\left(s \cdot 2^{i+2} - 1\right) 2^l} &\equiv \left(\frac{E_1^2 - 1}{4}\right)^{\left(s \cdot 2^{i+2} - 1\right) 2^l} && \mod 2 D_{12s \cdot 2^{l+i}}^\Gamma + v_1^j \cdot M_{12s \cdot 2^{l+i} - j}^\Gamma \\ &\equiv 0 && \text{if } j \leq 3 \cdot 2^l \end{aligned}$$

Proof. This immediately follows from (4) $\square$

Now let us treat the beta elements of order two, i.e. those with $i = 0$ in (1):

Proof of Theorem 1:
For part (i), we observe that:

$$\begin{aligned}
x_0^s &= v_2^s \\
&\equiv E_3^s && \bmod 2D_{3s}^\Gamma \\
&\equiv \left(E_3 - E_1^3\right)^s && \bmod 2D_{3s}^\Gamma + v_1 \cdot M_{3s-1}^\Gamma \\
&\equiv \left(\frac{E_1^2 - 1}{4}\right)^s && \bmod 2D_{3s}^\Gamma + v_1 \cdot M_{3s-1}^\Gamma
\end{aligned}$$

Similarly, for part (ii) we have:

$$\begin{aligned}
x_1^s &\equiv v_2^s && \bmod v_1^j \\
&\equiv E_3^{2s} && \bmod 2D_{6s}^\Gamma + v_1^j \cdot M_{6s-j}^\Gamma \\
&\equiv \left(E_3 - E_1^3\right)^{2s} && \bmod 2D_{6s}^\Gamma + v_1^j \cdot M_{6s-j}^\Gamma \\
&\equiv \left(\frac{E_1^2 - 1}{4}\right)^{2s} && \bmod 2D_{6s}^\Gamma + v_1^j \cdot M_{6s-j}^\Gamma
\end{aligned}$$

and since $j \leq a_{l+2} = 6 \cdot 2^l$ (and $j \leq 2^{l+2}$ if $s = 1$), for part (iii) we conclude:

$$\begin{aligned}
x_{2+l}^s &\equiv v_2^{4s \cdot 2^l} + v_1^{3 \cdot 2^l} v_2^{(4s-1)2^l} && \bmod \left(2, v_1^{a_{l+2}}\right) \\
&\equiv E_3^{4s \cdot 2^l} + E_3^{(4s-1)2^l} && \bmod 2D_{12s \cdot 2^l}^\Gamma + v_1^j \cdot M_{12s \cdot 2^l - j}^\Gamma \\
&\equiv \left(\frac{E_1^2 - 1}{4}\right)^{4s \cdot 2^l} + \left(\frac{E_1^2 - 1}{4}\right)^{(4s-1)2^l} && \bmod 2D_{12s \cdot 2^l}^\Gamma + v_1^j \cdot M_{12s \cdot 2^l - j}^\Gamma
\end{aligned}$$

In view of (13), this completes the proof. $\square$

Remark 2. Since $x_0 = v_2$ is sent to zero under the map (12) w.r.t. $(2, v_1)$, we see that in order to obtain something interesting, we have to impose $s \geq 3$ in part (i). In a similar vein, the condition $j \leq 2^{l+2}$ if $s = 1$ in part (iii) is needed to ensure that $D_{8s \cdot 2^l + j}^\Gamma \subset D_{12s \cdot 2^l}^\Gamma$ when using (13).

Next, we turn our attention to the elements $\beta_{4s \cdot 2^l/j, i+1}$ for $i \geq 1$:

Lemma 3. *For $l \geq 0$ and $i \geq 0$, we have*

$$x_{l+i+3} \equiv v_2^{2^{l+i+3}} + 2^{i+1} v_1^{3 \cdot 2^l} v_2^{\left(2^{i+3}-1\right)2^l} \quad \bmod \left(2^{i+2}, v_1^{a_{l+2}}\right)$$

Proof. Since $(a+b)^{2^{l+1}} \equiv a^{2^{l+1}} + b^{2^{l+1}} + 2(ab)^{2^l} \mod 4$ for $l \geq 0$, we compute

$$x_{l+3} = x_2^{2^{l+1}} \equiv v_2^{8 \cdot 2^l} + 2\left(v_1^3 v_2\right)^{2^l} v_2^{6 \cdot 2^l} \quad \bmod \left(4, v_1^{a_{l+2}}\right)$$

and use the binomial theorem. □

Proof of Theorem 2, part (i):
The choice $n = 2$ and $i = 1$ in (1) dictates $j = 2$, hence we compute

$$\begin{aligned} x_2^s &\equiv v_2^{4s} && \bmod \left(4, v_1^2\right) \\ &\equiv E_3^{4s} && \bmod 4D_{12s}^{\Gamma} + v_1^2 \cdot M_{12s-2}^{\Gamma} \\ &\equiv \left(\frac{E_1^2 - 1}{4}\right)^{4s} && \bmod 4D_{12s}^{\Gamma} + v_1^2 \cdot M_{12s-2}^{\Gamma} \end{aligned}$$

Combined with (13), this yields the claim. □

Proof of Theorem 2, part (ii):
In order to treat the remaining cases of our computation of $x_n^s \bmod \left(2^{i+1}, v_1^j\right)$, we notice that since (1) requires $j = m \cdot 2^i \leq a_{n-i}$, and since all cases with $i = 0$ and the case $i = 1$ for $n = 2$ have already been taken care of, it suffices to consider $n = l + i + 2$ where $l \geq 0$ and $i \geq 1$; now, for odd $s \geq 1$ we have (by Lemma 3 in a reindexed form)

$$\begin{aligned} x_{l+i+2}^s &\equiv v_2^{s \cdot 2^{l+i+2}} + 2^i v_1^{3 \cdot 2^l} v_2^{s \cdot 2^{l+i+2} - 2^l} && \bmod \left(2^{i+1}, v_1^{a_{l+2}}\right) \\ &\equiv E_3^{s \cdot 2^{l+i+2}} + 2^i E_3^{s \cdot 2^{l+i+2} - 2^l} && \bmod 2^{i+1} D_{12s \cdot 2^{l+i}}^{\Gamma} + v_1^j \cdot M_{12s \cdot 2^{l+i} - j}^{\Gamma} \end{aligned}$$

from which the desired result follows. □

Finally, we treat the beta elements defined by (2):

Proof of Theorem 2, part (iii):
In order to compute the f–invariant of $\beta_{4k/2,3}$, we are going to show that, although $z_{1,1} = v_1^2 - 4v_1^{-1} v_2$ is not holomorphic, we can still make sense out of the map (12) w.r.t. $(8, z_{1,1})$ if $t = 12k \geq 24$. To this end, we observe

$$v_1^6 = z_{1,1} v_1^4 + 4 v_1^3 v_2 = z_{1,1}\left(v_1^4 + 4 v_1 v_2\right) + 16 v_2^2,$$

hence we compute

$$\begin{aligned}
x_2^k &\equiv v_2^{4k} + k v_1^3 v_2^{4k-1} && \bmod \left(8, v_1^6\right) \\
&\equiv (1+4k)\, v_2^{4k} && \bmod \left(8, z_{1,1}\right) \\
&\equiv (1+4k)\, E_3^{4k} && \bmod 8D^\Gamma_{12k} + z_{1,1} M^\Gamma_{12k-2}
\end{aligned}$$

where $z_{1,1} M^\Gamma_{12k-2} \subset M^\Gamma_{12k}$ for dimensional reasons; finally, we note that

$$\begin{aligned}
E_3^{4k} &\equiv \left(\frac{E_1^2 - 1}{4}\right)^{4k} && \bmod 8D^\Gamma_{12k} + z_{1,1} M^\Gamma_{12k-2} \\
&\equiv \left(\frac{E_1^2 - 1}{4}\right)^{4k} v_1^4 z_{1,1} && \bmod 8D^\Gamma_{12k} + z_{1,1} M^\Gamma_{12k-2} \quad \text{if } k \geq 2
\end{aligned}$$

which completes the proof. □

Proof of Theorem 2, part (iv):
Recall that in the definition (2) we have to impose $j = m \cdot 2^i \leq a_{n-i-l}$ for $n \geq 3$; since the situation $m = i = 1$ has already been dealt with in the previous part (iii), it is sufficient to consider the case $n = l + i + 3$, $4 \leq m \cdot 2^i = j \leq a_{l+2}$, where $l \geq 0$, $i \geq 1$. In order to compute the f–invariants, we calculate the effect of the map (12) w.r.t. $(2^{i+2}, z_{i,m})$: Since

$$\begin{aligned}
v_1^{6\cdot 2^l} &= z_{i,m} v_1^{6\cdot 2^l - j} + 2j v_1^{6\cdot 2^l - 3} v_2 \\
v_1^{9\cdot 2^l} &= z_{i,m} \left(v_1^{9\cdot 2^l - j} + 2j v_1^{9\cdot 2^l - j - 3} v_2\right) + 4j^2 v_1^{9\cdot 2^l - 6} v_2^2
\end{aligned} \tag{16}$$

we calculate for $l \geq 0$, $i \geq 1$, and odd $s \geq 1$:

$$\begin{aligned}
x^s_{l+i+3} &\equiv v_2^{s\cdot 2^{l+i+3}} + 2^{i+1} v_1^{3\cdot 2^l} v_2^{\left(s 2^{i+3} - 1\right) 2^l} + \\
&\quad + 3s \cdot 2^i v_1^{6\cdot 2^l} v_2^{\left(s 2^{i+3} - 2\right) 2^l} && \bmod \left(2^{i+2}, v_1^{9\cdot 2^l}\right) \\
&\equiv v_2^{s\cdot 2^{l+i+3}} + 2^{i+1} v_1^{3\cdot 2^l} v_2^{\left(s 2^{i+3} - 1\right) 2^l} && \bmod \left(2^{i+2}, z_{i,m}\right)
\end{aligned}$$

hence

$$x^s_{l+i+3} \equiv E_3^{s\cdot 2^{l+i+3}} + 2^{i+1} E_3^{\left(s\cdot 2^{i+3} - 1\right) 2^l} \quad \bmod 2^{i+2} D^\Gamma_{24s\cdot 2^{l+i}} + z_{i,m} \cdot M^\Gamma_{24s\cdot 2^{l+i} - j}$$

and due to (16), application of Lemma 1 and Lemma 2 yields the claim. □

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